

Remarks on the speed of convergence of mixing coefficients and applications

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Abstract

In this paper, we study dependence coefficients for copula-based Markov chains. We provide new tools to check the convergence rates of copula-based Markov chains. We also provide a necessary condition for Markov chains from the Metropolis-hastings algorithm to be exponential ρ -mixing. A general necessary condition on symmetric copulas to generate exponential ρ -mixing is given. At the end of the paper, we comment and improve some of our previous results on mixtures of copulas.

Key words: Markov chains, copula, mixing conditions, reversible processes, ergodicity, Metropolis-hastings.

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1 Introduction

This work is motivated by a paper by Xiaohong Chen, Wei Biao Wu and Yanping Yi (2009) and a result of B.K. Beare (2010). In the paper [15], Xiaohong Chen and al. have shown that Markov processes generated by the Clayton, Gumbel or Student copulas are geometrically ergodic. They used in their paper quantile transformations and small sets to show geometric ergodicity, but could not handle for instance the mixture of these copulas. In the paper [7], M. Longla and al. have shown that these examples are actually exponential ρ -mixing. We have also answered the open question on geometric ergodicity of convex combinations of geometrically ergodic reversible Markov chains. In a recent paper, M. Longla (2012) obtained sufficient conditions for exponential ρ -mixing rate, improving a previous result of Beare (2010).

Quantifying the dependence among two or more random variables has been an enduring task for statisticians. Copulas are full measures of dependence among components of random vectors. Unlike marginal and joint distributions, which are directly observable, a copula is a hidden dependence structure that couples a joint distribution with its marginals. An early statistical application of

copulas was given by Clayton (1978) [3], where the dependence between two survival times in a multiple events study is modeled by the so-called Clayton copula

$$C(x, y) = (x^{-\alpha} + y^{-\alpha} - 1)^{-1/\alpha} \quad \alpha \geq 0.$$

The literature on copulas is growing fast; an excellent overview, guide to the literature and applications belongs to Embrechts and al. (2003) [5]. In later research into copulas, a driving force has been in financial risk management for modeling dependence among different assets in a portfolio. Nelsen's monograph (2006) [10] can be regarded as one of the best books for an introduction to copulas. We also define a new list of copula with functional parameters, that can be used in various model.

1.1 Definitions

1.1.1 2-Copulas

A 2-copula is a function $C : [0, 1] \times [0, 1] \rightarrow [0, 1] = I$ for which $C(0, y) = C(x, 0) = 0$ (C is grounded), $C(1, x) = C(x, 1) = x$ (each coordinate is uniform on I) and for all $[x_1, x_2; y_1, y_2] \subset I^2$, $C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) \geq 0$. The definition doesn't mention probability, but in fact these conditions imply that C is the joint cumulative distribution function of two random variables with margins uniform on I . It follows from the definition (see [10, chapter 1]) that the function C is non-decreasing in each of the parameters, has partial derivatives a.e with values between 0 and 1. The partial derivative of C are non-decreasing in the other parameter. A convex combination of 2-copulas is a 2-copula.

If X_1, X_2 are random variables with joint distribution F and marginal distributions F_1, F_2 , then the function $C(x, y)$ defined by $C(F_1(x_1), F_2(x_2)) = F(x_1, x_2)$ is a copula. Moreover, if the random variables are continuous, then the copula is uniquely defined by the joint distribution and the marginal distributions by the formula $F(F_1^{-1}(x_1), F_2^{-1}(x_2)) = C(x_1, x_2)$. This fact is known as Sklar's theorem. The implication of the Sklar's Theorem is that, after standardizing the effects of marginals, the dependence among components of X is fully described by the copula. Indeed, most conventional measures of dependence can be explicitly expressed in terms of the copula.

1.1.2 Copulas and Markov processes

Copulas have been shown to be a more flexible way to define a Markov process, as in the case when one suspects that the marginal distributions of the states are not related to the distribution of the initial state. Using copulas will allow changes in single marginal distributions, without having to change all other distributions in the chain.

A stationary Markov chain $(X_n, n \in \mathbb{Z})$ can be defined by a copula $(C(x, y))$ and a one dimensional marginal distribution. For stationary Markov chains with uniform marginals on $[0, 1]$, the transition probabilities for all $n \in \mathbb{Z}$ are $P(X_n \in A | X_{n-1} = x) = C_{,1}(x, y)$ for sets $A = (-\infty, y]$ (for more details, see [4, theorem 3.1]). Here, $C_{,1}(x, y)$ is the derivative of $C(x, y)$ with respect to x . This relationship was used in [15] to show that stationary Markov processes defined by the Clayton, Gumbel or Student copulas are geometrically ergodic. These notions will be defined later on. We will use in this paper the following conventional notation: $\|g\|_2^2 = \int_I g^2(x)dx$, $A_{,i}(x_1, x_2) = \frac{\partial A(x_1, x_2)}{\partial x_i}$, $c(x, y)$ will be used for the density of $C(x, y)$.

1.1.3 Dependence coefficients

Many dependence coefficients have been studied in the literature, such as α_n , β_n , ρ_n , ϕ_n among others. In our paper, we will mainly use the last 3 coefficients, that are defined as follows:

Given σ -fields \mathcal{A}, \mathcal{B} :

$$\begin{aligned}\beta(\mathcal{A}, \mathcal{B}) &= \mathbb{E} \sup_{B \in \mathcal{B}} |P(B|\mathcal{A}) - P(B)| \\ \rho(\mathcal{A}, \mathcal{B}) &= \sup_{f \in L^2(\mathcal{A}), g \in L^2(\mathcal{B})} \text{corr}(f, g), \\ \phi(\mathcal{A}, \mathcal{B}) &= \sup_{B \in \mathcal{B}, A \in \mathcal{A}, P(A) > 0} |P(B|A) - P(B)|\end{aligned}$$

Given the alternative form of the transition probabilities for a Markov chain generated by a copula and a marginal distribution with positive density, it was shown in [7] that these coefficients have the following simple form when the copula is absolutely continuous with density c , $\mathcal{A} = \sigma(X_i, i \leq 0)$ and $\mathcal{B} = \sigma(X_i, i \geq n)$:

$$\begin{aligned}\beta_n &= \int_0^1 \sup_{B \subset I} \left| \int_B (c_n(x, y) - 1) dy \right| dx, \\ \phi_n &= \sup_{B \subset I} \text{ess sup}_x \left| \int_B (c_n(x, y) - 1) dy \right|, \\ \rho_n &= \sup \left\{ \int_0^1 \int_0^1 c_n(x, y) f(x) g(y) dx dy : \|g\|_2 = \|f\|_2 = 1, \mathbb{E}(f) = \mathbb{E}(g) = 0 \right\}.\end{aligned}$$

Here, c_n is the density of the random variable (X_0, X_n) . In general the following inequalities hold (see [2, theorem 7.4, 7.5] for more.):

$$\beta_n \leq \phi_n, \quad \rho_n \leq 2\sqrt{\phi_n}, \quad \rho_n \leq (\rho_1)^n. \quad (1)$$

These coefficients are defined to assess the dependence structure of the Markov process, and provide necessary conditions for CLT and functional CLT and their rates of convergence for the statistics of the studied model. Some examples can be found in [11], [12] and the references therein. A stochastic process is said to be respectively an α -mixing (β -mixing or ρ -mixing), if $\alpha_n \rightarrow 0$ ($\beta_n \rightarrow 0$ or $\rho_n \rightarrow 0$). The process is exponentially mixing, if the convergence rate is exponential. A stochastic process is said to be geometrically ergodic, if for some $a \in (0, 1)$, and $n \in \mathbb{N}$,

$$\sup_{B \subset \mathcal{X}} |P(X_n \in B | X_0 = x) - P(X_n \in B)| \leq a^n W(x), \quad \text{with } \mathbb{E}(W(X_0)) < \infty.$$

By theorem 2.1 of Nemmelin and Tuominen (1982), *geometric ergodicity is equivalent to exponential β -mixing*. So, whenever we encounter the term, we may understand either of the two concepts. The main results on this topic are due to Darsow, Nguyen and Olsen (1992), de la Peña, Ibragimov and Sharakhmetov (2006) and Ibragimov (2009), who were among the first to study copula-based Markov processes. They characterized copula-based Markov chains. Joe (1997) proposed a class of parametric (strictly) stationary Markov models based on parametric copulas and parametric invariant distributions. Their setup of the problem was modified by Chen and Fan (2006), who studied a class of semi-parametric stationary Markov models based on parametric copulas and nonparametric invariant distributions and analyzed the strength of dependence in the Markov chain. They showed that the temporal dependence measure is fully determined by the properties of the copulas. Studying the strength of the dependence, Beare (2008) provided simple sufficient conditions for geometric β -mixing in terms of copulas without any tail dependence.

The tail dependent copulas are the most popular copulas because they suitably model shocks in economics. In an effort to address this issue, Ibragimov and Lentzas (2008) demonstrated via simulation that Clayton copula-based first-order strictly stationary Markov models could behave as “long memory” in copula levels (meaning that dependence on the initial distribution tends to persist). But, Xiaohong Chen, Wei Biao Wu and Yanqing Yi (2009) showed that this simulation is misleading, because the copula-based models studied by Ibragimov are actually geometric β -mixing. In conclusion, although a time series plot may look highly persistent and “long memory alike,” the time series may still be weakly dependent and “short memory ” so simulating isn’t a good way to handle the issue. Beare (2010) provided the proof based on simulation, that the Clayton copula provides exponential ρ -mixing. The theoretical proof of this fact and many others was given by M. Longla and M. Peligrad (2012) [7].

In this paper we derive the copula of the Metropolis-hastings algorithm for a general chosen proposal, and use one of our previous results to provide a condition for exponential ρ -mixing (or say a way to choose the proposal to guaranty exponential rate of convergence). We then have some remarks on the rate of convergence of Markov chains generated by convex combinations of copulas. These results are applied to the Frechet and Mardia families of copulas.

The paper is structured as follows: In section two below we provide new results on exponential ρ -mixing, and exponential β -mixing for some families of Metropolis-Hastings algorithms. In section 3 we comment and improve some of our results from [7] on convex combinations of copulas and apply them to some families of copulas.

2 Mixing rates of the Metropolis Hastings Algorithm

We shall recall that exponential ρ -mixing is equivalent to $\rho_1 < 1$ for Markov chains. This fact will be used in this section to improve some known results.

2.1 Introduction to the Metropolis Hastings algorithm

The M-H algorithm is one of the tools that statisticians use to simulate data from not completely known distributions; especially when the distributions are known up to the normalizing constant or when it is impossible to have i.i.d samples from the given distributions. The investigator is therefore looking for a Markov chain that converges to an invariant distribution equivalent to the incomplete one in hands. A sample from the stationary phase of this Markov chain is then used to study questions related to the initial unknown density. In order to achieve this goal, one should know the convergence rates of the Markov chain, that allows approximation of the start of the stationary phase of the chain. The mixing rates are important also for inference using statistics of the simulated data (they provide corresponding limit theorems for inference and testing). All these properties are highly related to the choice of the proposal for the simulation process. We will derive here the formula of the copula of these processes for some cases of proposals, that one can use along with other theorems in this dissertation to choose a proposal.

Let f be the invariant density we want to generate observations from using the M-H algorithm. Assume f is strictly positive on its domain with cumulative distribution function F . Let $q(x, y)$ be the proposal defined and positive on the domain of f (q is a density for each fixed value of x). Define,

$$\alpha(x, y) = \min \left\{ \frac{f(y)q(y, x)}{f(x)q(x, y)}, 1 \right\}.$$

Start the chain by a value in the domain of the distribution of interest, then sample potential new

states from the proposal distribution q which must be easy to sample from, and depends on the current state of the chain. More precisely, if the chain is initialized at $X_0 = x_0$, then at any time $t \geq 1$ the Metropolis-Hastings algorithm performs the following steps:

1. Generate an observation y from q , using x_{t-1} as parameter,
2. Compute the acceptance ratio $\alpha(x, y)$ defined above,
3. With probability $\alpha(x, y)$, set $X_t = y$ and otherwise $X_t = x_{t-1}$.

Implementation of step 3 requires generating an observation u from the uniform distribution on $[0, 1]$, then setting $X_t = y$ if $\alpha(x, y) \leq u$, otherwise $X_t = x_{t-1}$. It is therefore easy to derive the transition kernel of the obtained Markov chain:

$$P(x, dy) = q(x, y)\alpha(x, y)dy + (1 - \int q(x, y)\alpha(x, y)dy)\delta_x(dy). \quad (2)$$

We obtain an independent M-H algorithm by choosing an independent proposal (which is independent of the current state $q(y)$). Popular variants of the M-H algorithm are the random walk Metropolis (RWM) in which $y = x + \varepsilon$, where ε is generated from a spherically symmetric distribution, e.g., $\varepsilon \sim N(0; \Sigma)$, and the independent Metropolis (IM) defined above. Generally, the RWM is used in situations in which we have little idea about the shape of the target distribution and therefore we need to "totter" through the sample space. The opposite situation is one in which we have a pretty good idea about the target density and we are able to produce a credible approximation q that we use as proposal in the IM algorithm.

The speed and modes of convergence of the M-H chains have been studied by Mengersen and Tweedie (1996)[9], Roberts and Rosenthal [14], Jarner [6] and others. Despite understanding theoretically quite well the general convergence properties of the M-H algorithms, in practical implementations the user is left with the difficult task of determining and tuning an appropriate proposal distribution. For instance, in the case of a RWM one has to choose carefully the variance Σ so that a good balance is achieved between the acceptance rate and the chain's autocorrelation function. Similarly, in the case of an IM algorithm, one needs to find a distribution q that satisfies two non-trivial conditions:

- 1) it approximates reasonably well the target distribution and
- 2) it can be easily sampled from.

In practice, the process of determining a good proposal requires a back-and-forth strategy in which one starts with an initial proposal and subsequently makes a number of modifications while trying to assess their influence on the performance of the algorithm. This "tune-up" requires restarting the simulation process a number of times and can be time-consuming and is often frustrating. The difficulties are amplified when the target distribution has support in a high dimensional space. We propose here a way to avoid such problems, by choosing the "right" proposal to begin with. To simplify the formula of α , statisticians often use a symmetric proposal. For a general proposal $q(x, y)$, the following holds.

Proposition 1 *The copula of the Metropolis-Hastings algorithm with a proposal $q(x, y)$ and an invariant distribution f with cumulative distribution F is given by*

$$\begin{aligned} C(u, v) &= AC(u, v) + SC(u, v), \\ AC(u, v) &= \int_0^u \int_0^v \frac{q(F^{-1}(t), F^{-1}(z))}{f(F^{-1}(z))} \alpha(F^{-1}(t), F^{-1}(z)) dz dt \\ SC(u, v) &= \int_0^{\min\{u, v\}} \left[1 - \int_0^1 \frac{q(F^{-1}(t), F^{-1}(z))}{f(F^{-1}(z))} \alpha(F^{-1}(t), F^{-1}(z)) dz \right] dt. \end{aligned}$$

Proof. Using the definition of the copula from the Sklar's Theorem, and the relationship with the transition probability given in the Introduction, we obtain

$$C_{,1}(F(x), F(y)) = P(x, (-\infty, y]) = \int_{-\infty}^y P(x, dt).$$

Thus, using the chosen proposal and a simple change of variable, we obtain

$$C_{,1}(u, v) = \int_{-\infty}^{F^{-1}(v)} \alpha(F^{-1}(u), t) q(F^{-1}(u), t) dt + (1 - \int_{-\infty}^{\infty} \alpha(F^{-1}(u), t) q(F^{-1}(u), t) dt) \mathbb{I}(u < v)$$

Of course all this takes into account the fact that the given derivative acts like a distribution function (or in other words like Sobolev derivatives). Using simple facts from these theories, simple integration, the change of variable inside the integral ($z = F(t)$) and uniqueness of the copula lead to:

$$AC(u, v) = \int_0^u \int_0^v \frac{q(F^{-1}(t), F^{-1}(z))}{f(F^{-1}(z))} \alpha(F^{-1}(t), F^{-1}(z)) dz dt \quad (3)$$

$$SC(u, v) = \int_0^{\min\{u, v\}} [1 - \int_0^1 \frac{q(F^{-1}(t), F^{-1}(z))}{f(F^{-1}(z))} \alpha(F^{-1}(t), F^{-1}(z)) dz] dt. \quad (4)$$

■

AC is called the absolute continuous part of C and SC is the singular part of C . The copula is said to be absolutely continuous if $SC(u, v) = 0$ for all $(u, v) \in [0, 1]^2$.

For a strictly positive symmetric proposal $q(x, y)$, we have α simplifies and for the stated example of proposal we obtain:

$$AC(u, v) = \int_0^u \int_0^v \frac{h(F^{-1}(t))h(F^{-1}(z))}{f(F^{-1}(z))} \min\{\frac{f(F^{-1}(z))}{f(F^{-1}(t))}, 1\} dz dt \quad (5)$$

$$SC(u, v) = \int_0^{\min\{u, v\}} [1 - \int_0^1 \frac{h(F^{-1}(t))h(F^{-1}(z))}{f(F^{-1}(z))} \min\{\frac{f(F^{-1}(z))}{f(F^{-1}(t))}, 1\} dz] dt \quad (6)$$

In view of all these formulas we have the following:

Lemma 2 *The copula of the Metropolis-Hastings algorithm with a strictly positive symmetric proposal is absolutely continuous if and only if*

$$1 - \int_0^1 \frac{q(F^{-1}(t), F^{-1}(z))}{f(F^{-1}(z))} \min\{\frac{f(F^{-1}(z))}{f(F^{-1}(t))}, 1\} dz = 0 \quad \text{for almost all } t.$$

2.2 Mixing rates of the Metropolis-Hastings based on the proposal

As seen before, the choice of the proposal is crucial in the convergence of the MH algorithm. Here, we provide some conditions to measure ahead of time the rate of convergence for various classes of MH algorithms. The Independence Metropolis-Hastings algorithm is defined by $q(x, y) = q(y)$.

Theorem 3 *The Independence Metropolis-Hastings generates a geometrically ergodic and exponential ρ -mixing Markov chain if for almost all y , $q(y) \geq \beta f(y)$, $\beta \neq 0$.*

A related theorem was proved by Mengersen and Tweedie (1996)[9]. We provide here a proof of this statement based on our previous results from [8].

Theorem 4 *If there exists nonnegative functions $\varepsilon_1, \varepsilon_2$ defined on $[0, 1]$ for which the density of the absolute continuous part of the copula denoted $c(x, y)$ satisfies the inequality*

$$c(x, y) \geq \varepsilon_1(x) + \varepsilon_2(y)$$

with $\varepsilon_1, \varepsilon_2 \in L_1[0, 1]$ such that at least one of the two functions has a non-zero integral, then the process is exponential ρ -mixing, and thus exponential β -mixing if the density appears to be positive on a set of measure 1.

Proof. The density of the absolute continuous part of the copula of the metropolis-Hastings algorithm is

$$c(u, v) = \frac{q(F^{-1}(u), F^{-1}(v))}{f(F^{-1}(v))} \alpha(F^{-1}(u), F^{-1}(v)),$$

For the independence M-H algorithm with the given property, we obtain

$$c(u, v) \geq \frac{q(F^{-1}(v))}{f(F^{-1}(v))} \min\{\beta \frac{f(F^{-1}(v))}{q(F^{-1}(v))}, 1\} \geq \beta \min\{\beta \frac{f(F^{-1}(v))}{q(F^{-1}(v))}, 1\}.$$

Therefore, by Theorem 4 from [8], the chain is exponential ρ mixing. This chain being also absolutely regular (positive density of absolute continuous part of the copula on a set of measure 1), we can conclude that the chain is geometrically ergodic by Corollary in [7]. ■

A similar theorem holds for a non-symmetric proposal $q(x, y)$.

Theorem 5 *If the proposal $q(x, y)$ is such that $q(x, y) \geq \beta f(y)$ for all y , then the M-H is geometrically ergodic and exponential ρ -mixing.*

The proof is similar to the above one.

Remark 6 *Mengersen and Tweedie showed in their work [9] that the fact that without this bound on the ratio in the case of the independent algorithm the chain may tend to "stick" in regions of low density is of considerable practical importance and is not merely a curiosity: seemingly sensible procedures give this behavior. They provided an example to illustrate this fact.*

3 On mixing rates of mixtures copula-Based Markov chains

3.1 Convex combinations or mixtures of copulas

Mixing coefficients for copula-based Markov chains have been widely studied recently. We have provided some results based on properties of the copulas in recent papers. These results have some implications that we shall stress here.

Theorem 7

Let $(C_k(x, y); 1 \leq k \leq n)$ be n copulas. Then,

1. *if $C_1(x, y)$ generates exponential ρ -mixing, then any convex combination of these copulas that contains $C_1(x, y)$ generates exponential ρ -mixing stationary Markov chains.*

2. if one of the copulas in the combination generates an absolutely regular stationary Markov chain, and $C_1(x, y)$ exponential ρ -mixing stationary Markov chain, then any symmetric convex combination containing $C_1(x, y)$ generates geometrically ergodic and exponential ρ -mixing Markov chains.

Proof. The proof of the first part of the conclusion is similar to the proof of Theorem 5 of Longla and al. (2012) [7], except for the fact that we don't need symmetry of the copulas. Symmetry [7, in Theorem 5] was necessary only for the conclusion of geometric ergodicity. The proof of the second part need the reversibility condition on the convex combination, not on all copulas, because what matters is that the Markov chain that is generated by the convex combination is reversible. So, all arguments of [7, Theorem 5] work. ■

Remark 8 Basically, this theorem states that any convex combination of copulas that contains a copula that generates ρ -mixing will inherit this property. This happens no matter what are the mixing properties of other copulas in the combination. Moreover, if this convex combination is symmetric (which can happen even when the copulas themselves are not symmetric), then this combination will generate geometrically ergodic Markov chains provided that one of the copulas generates an absolutely regular Markov chain.

This is an implication of this result that can be useful in applications.

Corollary 9

Assume $(C_k(x, y); 1 \leq k \leq n)$ are n copulas and $C_1(x, y)$ has the density of its absolute continuous part strictly positive on a set of Lebesgue measure 1. Assume one of them generates a ρ -mixing stationary Markov chain. Then, any symmetric convex combination, $\sum_{k=1}^n a_k C_k(x, y)$ with $\sum_{k=1}^n a_k = 1$, $0 \leq a_i \leq 1$, $a_1 \neq 0$ generates a geometrically ergodic Markov chain.

3.2 Applications

1. The Mardia family of copulas

$$C(x, y) = \frac{\theta^2(1+\theta)}{2}M(x, y) + (1-\theta^2)P(x, y) + \frac{\theta^2(1-\theta)}{2}W(x, y), \quad \theta \in [-1, 1], \quad (7)$$

where the copulas W and M are the Hoeffding lower and upper bounds respectively and P is the independence copula. The formula (7) defines the Mardia family of copulas. This family of copulas is a convex combination of the copulas M , W and P .

2. The Frechet family of copulas

$$C(x, y) = \alpha M(x, y) + (1 - \alpha - \beta)P(x, y) + \beta W(x, y) \quad (0 \leq \alpha + \beta \leq 1). \quad (8)$$

The formula (8) defines the Frechet family of copulas. Notice that a Mardia copula is a Frechet copula with $\alpha + \beta = \theta^2$. So, they will share the same properties. Given that the copula P defines geometrically ergodic and ρ -mixing stationary Markov chains and these convex combinations are symmetric, we can conclude by the above Corollary 9 that these families generate exponential ρ -mixing and geometrically ergodic reversible Markov chains for $\alpha + \beta \neq 1$ ($\theta^2 \neq 1$) respectively.

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